STABILITY OF A LINEAR SYSTEM WITH RANDOM DISTURBANCES OF ITS PARAMETERS

(OB USTOICHIVOSTI LINEINOI SISTEMY PRI SLUCHAINYKH VOZMUSHCHENIIAKH EE PARAMETROV)

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The stability problem of a system described by an nth order equation with random coefficients is examined. Necessary and sufficient conditions of asymptotic stability in the mean-square are obtained. In the absence of noise these conditions transform into conditions of Routh and Hurwitz. Such sufficient conditions of moments of higher order are presented.

1. We assume that a certain determinate system is described by a linear differential equation of the order n with constant coefficients

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0 \qquad (1.1)$$

When random forces of the type of "white noise" act on such a system, Equation (1.1) transforms into the following stochastic differential equation

$$y^{(n)} + [a_1 + \eta_1^{\cdot}(t)]\, y^{(n-1)} + \ldots + [a_n + \eta_n^{\cdot}(t)]\, y = 0 \qquad (1.2)$$

It is assumed tnat Gaussian "white noises" $\xi_1^{\cdot}(t), \ldots, \xi_n^{\cdot}(t)$ have zero mathematical expectation but can, generally speaking, be correlated so that

$$M\eta_i^{\cdot}(t)\,\eta_j^{\cdot}(s) = 2a_{ij}\delta(t-s)$$

It is known that we can pass from noises $\eta_1^{\cdot}(t), \ldots, \eta_n^{\cdot}(t)$ to independent "white noises" $\xi_1^{\cdot}(t), \ldots, \xi_n^{\cdot}(t)$ with zero mathematical expectation and correlation matrix $2\delta_{ij}(t-s)$ by means of Equations

$$\eta_i^{\cdot}(t) = \sum_{j=1}^{n} \alpha_{ij}\xi_j^{\cdot}(t) \qquad (1.3)$$

where the matrix $\|\alpha_{ij}\|$ is such that

$$\|\alpha_{ij}\|\,\|\alpha_{ji}\| = \|a_{ij}^{*}\|$$

In the following, Equation (1.2) is considered as a system of stochastic differential equations of Itô (see, for example, [1], p.247) which taking into account (1.3) and introducing the notation

$$y = X_1,\ y' = X_2,\ \ldots,\ y^{n-1} = X_n$$

can be written in the form

$$dX_1 = X_2\,dt, \quad dX_2 = X_3\,dt, \ldots, dX_{n-1} = X_n\,dt$$

$$dX_n = -\sum_{i=1}^{n} a_i X_{n-i+1}\,dt - \sum_{i,\,j=1}^{n} \alpha_{ij} X_{n-i+1}\,d\xi_j(t) \qquad (1.4)$$

It is known that only one strictly Markov process exists with continuous trajectories $X^x(t) = (X_1^x(t), \ldots, X_n^x(t))$, which satisfy system (1.4) at the initial condition $X^x(0) = x$.

The process $X^x(t)$ is closely connected to the differential operator of second order

$$L = \sum_{i=1}^{n-1} x_{i+1} \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} a_i x_{n-i+1} \frac{\partial}{\partial x_n} + \Big(\sum_{i,j=1}^{n} a_{ij} x_{n-i+1} x_{n-j+1} \Big) \frac{\partial^2}{\partial x_n^2} \tag{1.5}$$

which in the investigation of stability of Markov processes plays the same role as the Liapunov operator in the stability of determinate systems (see [2 and 3]).

Following [2, 4 and 5], we say that system (1.4) is asymptotically p-stable ($p > 0$), if $\lim M|X^x(t)|^p = 0$ for $t \to \infty$ and in addition to this for any $\varepsilon > 0$ such a $\delta > 0$ is found that $M|X^x(t)| < \varepsilon$, if $|x| < \delta$ (here $|x|$ denotes the Euclidean norm of vector x). System (1.4) is called asymptotically stable in the mean-square if it is stable at $p = 2$.

A method for obtaining necessary and sufficient conditions of stability in the mean-square of an arbitrary linear system with "white noises" is indicated in the interesting paper [4], where a concept different from the one adopted in this paper for a linear stochastic system is examined. However, conditions obtained by this method are fairly cumbersome; for their verification it is necessary to compute n^2 determinants, the highest of which has the order n^2.

It is proven in [5] that for asymptotic stability in the mean-square of a stationary linear stochastic system it is necessary and sufficient that for any positive definite quadratic form $W(x)$ another positive definite quadratic form $V(x)$ be found for which $LV(x) = -W(x)$. This theorem permits to obtain (as was also noted in [2]) algebraic criteria of asymptotic stability in the mean-square for such a system. However, these criteria lead to even more cumbersome computations even in the determinate case.

In this paper necessary and sufficient conditions of stability in the mean-square are obtained for system (1.2) or (1.4). These conditions require computation of only $n+1$ determinants the highest of which has the order n (see (2.6) to (2.8)). In this connection it turns out that the first n determinants are the same as determinants Δ_k ($k = 1, 2, \ldots, n$) which enter into the criterion of Routh-Hurwitz for Equation (1.1). The last determinant, however, is obtained by exchanging the first row in Δ_n by a row which is composed according to a definite rule from coefficients a_{ij} of the correlation matrix. If all $a_{ij} = 0$ then criterion (2.6) to (2.8) transforms into criterion of Routh-Hurwitz.

It is also interesting to note that in the special case when all a_{ij} are equal to zero with the exception of a single one, it follows from the criterion obtained that the necessary and sufficient condition presented for this case in [6] can be significantly simplified. For $n = 2$, $a_{11} = a_{12} = a_{21} = 0$ this result coincides with the result of [7].

2. It was shown in [5] that for asymptotic stability in the mean-square of system (1.4) it is necessary that the system "without randomness"

$$dX_1 = X_2\, dt, \quad dX_2 = X_3\, dt, \ldots, \quad dX_{n-1} = X_n dt, \quad dX_n = -\sum_{i=1}^{n} a_i X_{n-i+1} \tag{2.1}$$

be asymptotically stable, i.e. that conditions of Routh-Hurwitz be fulfilled

$$\Delta_1 = a_1 > 0, \quad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} > 0, \quad \Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0, \ldots, \Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ 1 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ 0 & 1 & a_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix} > 0$$

It is known that when these conditions are fulfilled, there exists a positive definite quadratic form $V(x)$ for which by virtue of (2.1) the total derivative

$$L_0V = \sum_{i=1}^{n-1} x_{i+1} \frac{\partial V}{\partial x_i} - \sum_{i=1}^{n} a_i x_{n-i+1} \frac{\partial V}{\partial x_n} \tag{2.2}$$

represents a preassigned negative definite form.

We assume initially that the quadratic form

$$a(x) = \sum_{i,j=1}^{n} a_{ij} x_{n-i+1} x_{n-j+1} \qquad (a_{ij} = a_{ji})$$

is positive definite. Then the following is valid.

L e m m a 2.1 . For asymptotic stability in the mean-square of system (1.4) it is necessary and sufficient for a positive definite quadratic form

$$V(x) = \sum_{i,j=1}^{n} d_{ij} x_i x_j$$

to exist and satisfy the conditions

$$L_0V(x) = -a(x), \qquad d_{nn} < 1/2 \tag{2.3}$$

P r o o f . In fact, let the quadratic form $V(x)$ exist with coefficients d_{ij} and satisfy the conditions of the Lemma. By virtue of (1.5), (2.2) and (2.3)

$$LV = L_0V + a(x) \frac{\partial^2 V}{\partial x_n^2} = (2d_{nn} - 1)\, a(x) < 0$$

According to the already previously mentioned theorem of [5] it follows from this that system (1.4) is asymptotically stable in the mean-square.

On the other hand in case of asymptotic stability of system (1.4) according to the same theorem, there exists a positive definite quadratic form

$$V_1(x) = \sum_{i,j=1}^{n} e_{ij} x_i x_j$$

for which $LV_1(x) = -a(x)$, i.e.

$$L_0V_1 = LV - a(x) \frac{\partial^2 V}{\partial x_n^2} = -(2e_{nn} + 1)\, a(x)$$

In this manner $V = V_1(x)/(2e_{nn} + 1)$ and consequently

$$d_{nn} = e_{nn} / (2e_{nn} + 1) < 1/2.$$

Lemma is proved.

For obtaining the desired conditions it is sufficient to express the coefficient d_{nn} in the form $V(x)$ which is to be determined from Equation (2.3) through parameters a_i and a_{ij} of system (1.4).

For this purpose we denote by $\overset{\circ}{X}_{1j}(t), \ldots, \overset{\circ}{X}_{nj}(t)$ $(j = 1, 2, \ldots, n)$ the fundamental system of solutions of determinate equations (2.1). This system is defined through initial conditions $\overset{\circ}{X}_{sj}(0) = \delta_{sj}$. Then any solution $X_1^{\circ x}(t), \ldots, X_n^{\circ x}(t)$ of these equations with initial conditions $X_i^{\circ x}(0) = x_i$ $(i = 1, 2, \ldots, n)$ is written on the form

$$X_i^{\circ x}(t) = \sum_{j=1}^{n} x_j \overset{\circ}{X}_{ij}(t)$$

It is known [8] that function $V(x)$ which satisfies relationship (2.3) can be represented in the form

$$V(x) = \int_0^\infty \sum_{i,j=1}^{n} a_{ij} X_{n-i+1}^{\circ x}(u)\, X_{n-j+1}^{\circ x}(u)\, du$$

The last equation permits to express coefficients d_{ij} of form $V(x)$ and in particular coefficient d_{nn} through the fundamental system of solutions $\overset{\circ}{X}_{ij}(t)$, and subsequently, as was shown in [9], through coefficients a_i and a_{ij} . In fact it follows from [9] that

$$d_{nn} = \frac{1}{2\Delta_n} \sum_{r=0}^{n-1} q_{nn}^{(r)} \Delta_{1, r+1} \tag{2.4}$$

where $\Delta_{1,r+1}$ is the cofactor of the element of the first row and $r+1$ column of the last determinant of Hurwitz Δ_n, while quantities $q_{nn}^{(r)}$ are related to coefficients a_{ij} of form $a(x)$ by Equation

$$(-1)^{n-1} \sum_{i,j=1}^{n} a_{n-i+1,\, n-j+1} D_{ni}(\lambda) D_{nj}(-\lambda) = \sum_{r=0}^{n-1} q_{nn}^{(r)} \lambda^{(n-r-1)} \tag{2.5}$$

Here $D_{nj}(\lambda)$ is the cofactor of the element of the nth row and jth column of determinant $D(\lambda)$ of system (1.4)

$$D(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 & \ldots & 0 \\ 0 & -\lambda & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & a_1 - \lambda \end{vmatrix}$$

It is easy to see that $D_{ni}(\lambda) D_{nj}(-\lambda) = (-1)^{i+j-1}\lambda^{j-2}$. Therefore we obtain from (2.5)

$$\sum_{k=0}^{n-1} \Big(\sum_{p+q=2(n-k)} (-1)^{q+1} a_{pq} \Big) \lambda^{2k} = \sum_{k=0}^{n-1} q_{nn}^{(n-k-1)} \lambda^{2k}, \qquad q_{nn}^{(n-k-1)} = \sum_{p+q=2(n-k)} (-1)^{q+1} a_{pq} \tag{2.6}$$

From Lemma 2.1, (2.4) and (2.6) it follows that in the case where $a(x)$ represents a positive definite quadratic form it is necessary and sufficient for stability in the mean-square of system (1.4) that the following conditions be fulfilled

$$\Delta_1 > 0, \ldots, \Delta_n > 0, \ \Delta_n > \Delta \tag{2.7}$$

Here

$$\Delta = \begin{vmatrix} q_{nn}^{(0)} & q_{nn}^{(1)} & q_{nn}^{(2)} & \ldots & q_{nn}^{(n-1)} \\ 1 & a_2 & a_4 & \ldots & 0 \\ 0 & a_1 & a_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a_n \end{vmatrix} \tag{2.8}$$

differs from the last determinant of Hurwitz Δ_n only in the first row. In this connection quantities $q_{nn}^{(r)}$ $(r = 0, 1, \ldots, n-1)$ are expressed through coefficients a_{ij} of the correlation matrix according to Equations (2.6).

It will be shown now that conditions (2.6) to (2.8) remain in force even without the assumption regarding the positive definite character of the quadratic form $a(x)$. For this purpose another system, together with (1.4), will be examined

$$dX_1 = X_2\, dt, \quad dX_2 = X_3\, dt, \ldots, dX_{n-1} = X_n\, dt \tag{2.9}$$

$$dX_n = -\sum_{i=1}^{n} a_i X_{n-i+1}\, dt - \sum_{i,j=1}^{n} a_{ij} X_{n-i+1}\, d\xi_j + \varepsilon X_1\, d\eta_1 + \varepsilon^2 \sum_{i=2}^{n} X_i\, d\eta_i$$

Here $\eta_1(t), \ldots, \eta_n(t)$ are Wiener processes independent of each other and of $\xi_1(t), \ldots, \xi_n(t)$; ε is a small parameter.

It is easy to see that the operator which corresponds to system (2.9) has the form

$$L_\varepsilon = L + \Big(\varepsilon^2 x_1^2 + \sum_{i=2}^{n} \varepsilon^4 x_i^2 \Big) \frac{\partial^2}{\partial x_n^2}$$

Since the quadratic form

$$a_\varepsilon(x) = a(x) + \varepsilon^2 x_1^2 + \varepsilon^4 \sum_{i=2}^{n} x_i^2$$

is positive definite for any $\varepsilon > 0$, it is necessary and sufficient for asymptotic stability in the mean-square of system (2.9) that the following conditions be satisfied

$$\Delta_1 > 0, \ldots, \Delta_n > 0, \qquad \Delta_n > \Delta_\varepsilon$$

Here

$$\Delta_\varepsilon = \begin{vmatrix} q_{nn}^{(0)} + \varepsilon^4 q_{nn}^{(1)} - \varepsilon^4 q_{nn}^{(2)} + \varepsilon^4 \ldots (-1)^{n-1}(a_{nn} + \varepsilon^2) \\ 1 & a_2 & a_4 & \ldots & 0 \\ 0 & a_1 & a_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a_n \end{vmatrix} =$$

$$= \Delta + (-1)^{n-1} \varepsilon^2 \Delta_{1n} + \varepsilon^4 \sum_{i=1}^{n-1} (-1)^{i-1} \Delta_{1i} \tag{2.10}$$

Since for $(-1)^{n-1} \Delta_{1n} > 0$, it follows from (2.10) that for all sufficiently small ε

$$\Delta_\varepsilon > \Delta \tag{2.11}$$

Now we shall assume that system (1.4) is asymptotically stable in the mean-square. Then according to theorem 5.2 of [5], system (2.9) will also be stable for all sufficiently small $\varepsilon > 0$ and therefore

$$\Delta_1 > 0, \ldots, \Delta_n > 0, \qquad \Delta_n > \Delta_\varepsilon \tag{2.12}$$

From (2.11) and (2.12) we have

$$\Delta_1 > 0, \ldots, \Delta_n > 0, \qquad \Delta_n > \Delta \tag{2.13}$$

Conversely, let inequalities (2.13) be satisfied. Then it follows from (2.10) that a sufficiently small ε can be found such that inequalities (2.12) are satisfied, i.e. system (2.9) is asymptotically stable in the mean-square for this ε. Consequently, (see [5], Theorem 5.1) system (1.4) is also asymptotically stable in the mean-square. Therefore the following theorem is valid.

T h e o r e m 2.1 . For asymptotic stability in the mean-square of system (1.4) it is necessary and sufficient that the following conditions be fulfilled

$$\Delta_1 > 0, \ldots, \Delta_n > 0, \qquad \Delta_n > \Delta \tag{2.14}$$

Here determinant Δ has the form (2.8), while quantities $q_{nn}^{(r)}$ ($r = 0, 1, \ldots, n-1$) in the first row of this determinant are expressed through coefficients a_{ij} according to Equations (2.6).

It is noted that only those coefficients a_{ij} of the correlation matrix enter into conditions (2.14) for which the sum $i + j$ is even. In particular, for systems of second and third order necessary and sufficient conditions of asymptotic stability in the mean-square have the form for $n = 3$.

$$a_1 > 0, \quad a_2 > 0, \quad a_1 a_2 > a_{11} a_2 + a_{22} \qquad \text{for } n = 2$$

$$a_1 > 0, \quad a_3 > 0, \quad a_1 a_2 > a_3, \quad (a_1 a_2 - a_3) a_3 > a_{11} a_2 a_3 + a_{33} a_1 + a_3 (a_{22} - 2 a_{13})$$

In the case where independent white noises $\eta_1^{\cdot}, \ldots, \eta_n^{\cdot}$ are added to coefficients a_i of Equation (1.1), i.e. $a_{ij} = 0$ for $i \neq j$, the determinant Δ assumes the most simple form

$$\Delta = \begin{vmatrix} a_{11} & -a_{22} & a_{33} & \ldots & (-1)^{n-1} a_{nn} \\ 1 & a_2 & a_4 & \ldots & 0 \\ 0 & a_1 & a_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a_n \end{vmatrix}$$

3. Conditions (2.6) to (2.8) obtained in Section 2, are sufficient for asymptotic p-stability of the system (1.4) for $p \le 2$. In this section we shall present sufficient conditions for asymptotic p-stability for $p > 2$. It is assumed initially that the quadratic form $a(x)$ is positive definite.

For asymptotic p-stability at $p = 2$, and therefore also at $p > 2$, it is necessary for the following positive definite quadratic form to exist

$$V(x) = \sum_{i,\, j=1}^{n} d_{ij} x_i x_j \qquad (d_{ij} = d_{ji})$$

and to satisfy relationships (2.3). We set

$$V^\circ(x) = [V(x)]^{1/2 p}$$

It is not difficult to see that

$$LV^\circ = pV^{1/2 p-2}\left\{\frac{VL_0V}{2} + a(x)\left[Vd_{nn} + (p-2)\Big(\sum_{j=1}^{n} d_{nj}x_j\Big)^2\right]\right\} =$$

$$= pV^{1/2 p-2}\, a(x)\left[V(d_{nn} - 1/2) + (p-2)\Big(\sum_{j=1}^{n} d_{nj}x_j\Big)^2\right] \tag{3.1}$$

From the known inequality for the positive definite self-adjoint matrix [10]

$$(D_{x,y}) \leqslant (D_{x,x})\,(D_{y,y})$$

for $y = (0,\ldots,0,\,1)$ it follows that

$$\Big(\sum_{j=1}^{n} d_{nj}x_j\Big)^2 \leqslant d_{nn} V(x)$$

Making use of this relationship we obtain from (3.1)

$$LV^\circ \leqslant pV^{1/2 p-1}\, a(x)\,[d_{nn}(p-1) - 1/2] \tag{3.2}$$

If $d_{nn}(p-1) < \frac{1}{2}$ then it follows from Theorem 2.2 of paper [5] and (3.2) that system (1.4) is asymptotically p-stable.

In this manner for asymptotic stability of system (1.4) for $p > 2$ it is sufficient that the following inequalities be satisfied. The first n of these inequalities are necessary

$$\Delta_1 > 0, \ldots, \Delta_n > 0, \qquad \Delta_n > (p-1)\,\Delta \tag{3.3}$$

We can present examples which show that condition $\Delta_n > (p-1)\Delta$ is not necessary.

It is not difficult in a manner analogous to Section 2 to show that conditions (3.3) remain valid even without the assumption regarding nondegeneracy of the quadratic form $a(x)$

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